

The Mathematics Of Stock Option Valuation - Part Five

Deriving The Black-Scholes Model Via Risk-Neutral Probabilities

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In Part One we explained why valuing a call option as a stand-alone asset using risk-adjusted discount rates will almost always lead to an incorrect value because the value determined in this manner will most likely be subject to arbitrage. In Part Two we calculated the no-arbitrage price of a call option in the one-period economy via partial differential equations. In Part III we calculated the no-arbitrage price of a call option in the one-period economy via risk-neutral probabilities. In Part IV we switched from the one-period economy to continuous time where we derived the Black-Scholes option pricing model via partial differential equations. In this section we will stay in continuous time and will derive the Black-Scholes model via risk-neutral probabilities.

Legend of Symbols

C_t	=	Call option price at time t
R_t	=	Return on the stock at time t
S_t	=	Stock price at time t
K	=	Call option exercise price
P	=	The actual probability measure
Q	=	The risk-neutral probability measure
T	=	Time to option expiration in years
r	=	Annual risk-free rate of interest
t	=	Current time period in years
z	=	Normally-distributed random variable with mean zero and variance one
μ	=	Annual expected rate of return on the stock
σ	=	Annual standard deviation of returns (volatility)
W_t	=	Brownian motion with mean zero and variance t
$N[z]$	=	The cumulative normal distribution function of the random variable z

Introduction

Under the actual probability measure the expected stock price at time T discounted at the risk-free rate is a sub-martingale, which means that the discounted value is greater than the stock price at time t where $t < T$. This is so because given that investors are risk-averse they demand a premium over-and-above the risk-free rate for taking equity risk. The mathematical equivalent of this relationship is...

$$\mathbb{E}^P \left[e^{-r(T-t)} S_T \right] > S_t \quad (1)$$

Under the risk-neutral probability measure the expected stock price at time T discounted at the risk-free rate is a martingale, which means that the discounted value is equal to the stock price at time t where $t < T$. This is so because even though the risk-neutral probability measure Q maintains the same variance as the actual probability measure P it's mean is shifted to the left such that more weight is placed on negative outcomes and less weight is placed on positive outcomes. Under the risk-neutral measure the investor's risk-premium is embedded in the risk-adjusted probabilities. The mathematical equivalent of this relationship is...

$$\mathbb{E}^Q \left[e^{-r(T-t)} S_T \right] = S_t \quad (2)$$

The Fundamental Theorem of Finance states that in a complete market all asset prices discounted at the risk-free rate are martingales under the unique risk-neutral probability measure. The value of our call option at time t is therefore the expected call payoff at time T under the risk-neutral probability measure discounted at the risk-free rate. The mathematical equivalent of this relationship is...

$$C_t = \mathbb{E}^Q \left[e^{-r(T-t)} \text{Max} \left\{ S_T - K, 0 \right\} \right] \quad (3)$$

We will derive the Black-Scholes equation via the following steps:

- Step 1** - Derive the equation for stock price under the actual probability measure P .
- Step 2** - Derive the equation for stock price under the risk-neutral probability measure Q .
- Step 3** - Derive an equation for call price via the risk-neutral probability measure derived in Step 2 above.
- Step 4** - Derive the Black-Scholes option pricing equation.

Step 1 - Equation For Stock Price Under The Actual Probability Measure

We will model stock price as a continuous time stochastic process. The equation for the change in stock price at any time t as a function of a deterministic return and an innovation is...

$$\delta S_t = S_t \mu \delta t + S_t \sigma \delta W_t \quad (4)$$

Equation (4) is referred to as a stochastic differential equation (SDE). The variable μ is the expected rate of return under the actual probability measure P and consists of the risk-free rate plus a premium for risk. The variable δW_t is the change in the driving Brownian motion and is normally-distributed with mean zero and variance δt . The equation for stock price at call expiration under the actual probability measure P is...

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma W_T} \quad (5)$$

Equation (5) is the solution to the SDE in equation (4). The variable W_T is the driving Brownian motion and is a random variable. At time t the value of the Brownian motion W_t is zero. At time T the value of the Brownian motion W_T is normally-distributed with mean zero and variance $T - t$. By definition...

$$W_T \sim N[0, T - t] \text{ such that } \mathbb{E}[W_T] = 0 \text{ and } \mathbb{E}[W_T^2] = T - t \quad (6)$$

We will replace the innovation term σW_T in equation (5) above with its equivalent of $\sigma \sqrt{T-t} z$ where z is a normally distributed random variable with mean zero and variance one. The normalized equation for stock price is...

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t} z} \quad (7)$$

Using the normalized stock price equation (7) above the equation for expected stock price at time T under the actual probability measure P discounted at the risk-free rate is...

$$\begin{aligned} \mathbb{E}^P \left[e^{-r(T-t)} S_T \right] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t} z} \delta z \\ &= S_t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 + \sigma \sqrt{T-t} z - \frac{1}{2}\sigma^2(T-t)} e^{(\mu-r)(T-t)} \delta z \\ &= S_t e^{(\mu-r)(T-t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - \sigma \sqrt{T-t})^2} \delta z \\ &= S_t e^{(\mu-r)(T-t)} \end{aligned} \quad (8)$$

Note that the last integral in equation (8) above is the cumulative normal distribution function of a random variable (with mean $\sigma \sqrt{T-t}$ and variance one) and integrates to one.

The discounted stock price under the actual probability measure P is $S_t e^{(\mu-r)(T-t)}$, which is not a martingale since for risk-averse investors $\mu > r$. In the next section we will make the discounted stock price a martingale by switching from the actual probability measure P to the risk-neutral probability measure Q .

Step 2 - Equation For Stock Price Under The Risk-Neutral Probability Measure

The variable W_t in equation (4) above is the driving Brownian motion for stock price under the actual probability measure. We will define a new Brownian motion \tilde{W}_t that will operate under the risk-neutral probability measure. Both Brownian motions are zero at time $t = 0$ and have the same variance t . The mathematical relationship between these two Brownian motions is...

$$\delta\tilde{W}_t = \delta W_t + \alpha\delta t \quad (9)$$

Or equivalently...

$$\delta W_t = \delta\tilde{W}_t - \alpha\delta t \quad (10)$$

Under the actual probability measure P the stock earns a rate of return equal to μ . Under this probability measure the expected value of the stock at time T discounted at the risk-free rate is not a martingale. Under the risk-neutral probability measure we want the expected value of the stock at time T discounted at the risk-free rate to be a martingale. To do this the stock must earn the risk-free rate under measure Q . We will define return as equation (4) divided by stock price. The equation for return is...

$$R_t = \frac{\delta S_t}{S_t} = \mu\delta t + \sigma\delta W_t \quad (11)$$

The following equation transforms the return on the stock under the actual probability measure P to the return under the risk-neutral probability measure Q . We do this by substituting equation (10) for δW_t in equation (11) above. The equation for return under the risk-neutral measure is...

$$\begin{aligned} R_t &= \mu\delta t + \sigma\delta W_t \\ &= \mu\delta t + \sigma\left\{\delta\tilde{W}_t - \alpha\delta t\right\} \\ &= \mu\delta t - \alpha\sigma\delta t + \sigma\delta\tilde{W}_t \\ &= (\mu - \alpha\sigma)\delta t + \sigma\delta\tilde{W}_t \end{aligned} \quad (12)$$

The mean of the return distribution (see Appendix C) is...

$$mean = \mathbb{E}\left[R_t\right] = (\mu - \alpha\sigma)\delta t \quad (13)$$

The variance of the return distribution (see Appendix D) is...

$$variance = \mathbb{E}\left[R_t^2\right] - \left[\mathbb{E}\left[R_t\right]\right]^2 = \sigma^2\delta t \quad (14)$$

Note that the expected return under the actual probability measure P is $\mu\delta t$ and under the risk-neutral probability measure it is $\mu\delta t - \alpha\sigma\delta t$. Also note that the variance of both probability distributions is $\sigma^2\delta t$. What we have done is shifted the actual probability distribution to the left thus putting more weight on negative outcomes and less weight on positive outcomes while keeping the shape of the distribution (i.e. variance) unchanged. Because we have shifted the actual probability distribution to the left the investor's risk premium under the risk-neutral probability measure is zero.

We want the expected return under the risk-neutral probability measure to be the risk-free rate. We do this by equating the expectation in equation (13) above to the risk-free rate and solve for alpha. For the stock price to be a martingale under the risk-neutral measure the value of alpha must be...

$$\begin{aligned} (\mu - \alpha\sigma)\delta t &= r\delta t \\ \mu - \alpha\sigma &= r \\ \alpha &= \frac{\mu - r}{\sigma} \end{aligned} \quad (15)$$

The equation for the change in stock price at time t as a function of a deterministic return and an innovation under the new probability measure \tilde{W}_t is...

$$\delta S_t = S_t \mu \delta t + S_t \sigma \delta W_t \quad (16)$$

$$\begin{aligned} &= S_t \mu \delta t + S_t \sigma \left[\delta \tilde{W}_t - \alpha \delta t \right] \\ &= S_t \mu \delta t + S_t \sigma \left[\delta \tilde{W}_t - \left\{ \frac{\mu - r}{\sigma} \right\} \delta t \right] \\ &= S_0 r \delta t + S_0 \sigma \delta \tilde{W}_t \end{aligned} \quad (17)$$

The equation for stock price at time T under the risk-neutral probability measure where $\tilde{W}_t = 0$ is...

$$S_T = S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma \tilde{W}_T} \quad (18)$$

After normalizing the random variable \tilde{W}_T such that it has mean zero and variance one the equation for stock price becomes...

$$S_T = S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t} z} \quad (19)$$

The expected value of the stock price at time t under the risk-neutral probability measure discounted at the risk-free rate is...

$$\begin{aligned} \mathbb{E}^Q \left[e^{-r(T-t)} S_T \right] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} S_T \delta z \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t} z} \delta z \\ &= S_t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 + \sigma \sqrt{T-t} z - \frac{1}{2}\sigma^2(T-t)} \delta z \\ &= S_t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - \sigma \sqrt{T-t})^2} \delta z \\ &= S_t \end{aligned} \quad (20)$$

Note that the last integral in equation (20) above is the cumulative normal distribution function of a random variable (with mean $\sigma \sqrt{T-t}$ and variance one) and integrates to one.

Under the risk-neutral measure the discounted value of expected stock price via equation (20) is a martingale.

Step 3 - Derive an Equation for Call Price

We will combine equations (3) and (20) above to value the call option. The value of the call option at time t under the risk-neutral probability measure Q is...

$$\begin{aligned} C_t &= \mathbb{E}^Q \left[e^{-r(T-t)} \text{Max} \left\{ S_T - K, 0 \right\} \right] \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} \text{Max} \left\{ S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t} z} - K, 0 \right\} \delta z \end{aligned} \quad (21)$$

We will define \hat{z} as the minimum value of the random variable z in equation (21) above such that the call option is in the money (i.e. stock price \geq exercise price). By changing the lower bound of integration we can remove the max function. The equation for call price becomes...

$$C_t = \int_{\hat{z}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} \left[S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma \sqrt{T-t} z} - K \right] \delta z \quad (22)$$

We want to be able to use the cumulative normal distribution function, which is an integral where the lower bound of integration is negative infinity and the upper bound of integration is less than infinity. The normal distribution function has various symmetry properties one of which is...

$$\int_{\hat{z}}^{\infty} f(u)\delta u = \int_{-\infty}^{-\hat{z}} f(u)\delta u \quad (23)$$

Because we want to use the cumulative normal distribution function we will make the above transformation. One item to note is that in making this transformation the sign of $\sigma\sqrt{T-t}$ will change to $-\sigma\sqrt{T-t}$. The equation for call price becomes...

$$C_t = \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} \left[S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)-\sigma\sqrt{T-t}z} - K \right] \delta z \quad (24)$$

We can break the integral in equation (24) above into two parts, A and B, so that we can evaluate them separately. The equation for call price becomes...

$$\begin{aligned} C_t &= \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)-\sigma\sqrt{T-t}z} \delta z - \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-r(T-t)} K \delta z \\ &= S_t \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 - \sigma\sqrt{T-t}z - \frac{1}{2}\sigma^2(T-t)} \delta z - K e^{-r(T-t)} \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \delta z \end{aligned} \quad (25)$$

The value of the first integral is...

$$\begin{aligned} A &= S_t \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 - \sigma\sqrt{T-t}z - \frac{1}{2}\sigma^2(T-t)} \delta z \\ &= S_t \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z+\sigma\sqrt{T-t})^2} \delta z \end{aligned} \quad (26)$$

We will define $\theta = z + \sigma\sqrt{T-t}$ and make this substitution in equation (26) above. Note that in making this substitution we must change the bounds of integration such that the equation for A becomes...

$$\begin{aligned} A &= S_t \int_{-\infty}^{-\hat{z}+\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta^2} \delta\theta \\ &= S_t N[-\hat{z} + \sigma\sqrt{T-t}] \end{aligned} \quad (27)$$

The value of the second integral is...

$$\begin{aligned} B &= K e^{-r(T-t)} \int_{-\infty}^{-\hat{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \delta z \\ &= K e^{-r(T-t)} N[-\hat{z}] \end{aligned} \quad (28)$$

The value of the call option at time t is therefore...

$$C_t = A - B = S_t N[-\hat{z} + \sigma\sqrt{T-t}] - K e^{-r(T-t)} N[-\hat{z}] \quad (29)$$

Step 4 - Derive the Black-Scholes Equation

We will now derive the Black-Scholes equation:

We will first solve for $-\hat{z}$, which is the point at which the call option is in-the-money. The value of $-\hat{z}$ is...

$$\begin{aligned}
 S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}(-\hat{z})} &= K \\
 (r - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}(-\hat{z}) &= \ln K - \ln S_t \\
 -\hat{z} &= \frac{\ln K - \ln S_t - (r - \frac{1}{2}\sigma^2)(T-t)}{-\sigma\sqrt{T-t}} \\
 -\hat{z} &= \frac{\frac{\ln S_t}{\ln K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
 \end{aligned} \tag{30}$$

We will define $d1$ to be...

$$\begin{aligned}
 d1 &= -\hat{z} + \sigma\sqrt{T-t} \\
 &= \frac{\frac{\ln S_t}{\ln K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} + \sigma\sqrt{T-t} \\
 &= \frac{\frac{\ln S_t}{\ln K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
 \end{aligned} \tag{31}$$

We will define $d2$ to be...

$$\begin{aligned}
 d2 &= -\hat{z} \\
 &= d1 - \sigma\sqrt{T-t}
 \end{aligned} \tag{32}$$

Our equation (29) for call price at time t using the above definitions is the Black-Scholes model. The equation for call price is...

$$C_t = S_t N(d1) - K e^{-r(T-t)} N(d2) \tag{33}$$

Appendix

A) The first moment of the distribution of R_t as defined in equation (12) above is...

$$\begin{aligned}
 \mathbb{E}[R_t] &= \mathbb{E}\left[(\mu - \alpha\sigma)\delta t + \sigma\delta\tilde{W}_t\right] \\
 &= \mathbb{E}\left[(\mu - \alpha\sigma)\delta t\right] + \mathbb{E}\left[\sigma\delta\tilde{W}_t\right] \\
 &= (\mu - \alpha\sigma)\delta t + \sigma\mathbb{E}\left[\delta\tilde{W}_t\right] \\
 &= (\mu - \alpha\sigma)\delta t
 \end{aligned} \tag{34}$$

B) The second moment of the distribution of R_t as defined in equation (12) above is...

$$\begin{aligned}
 \mathbb{E}[R_t^2] &= \mathbb{E}\left[\left\{(\mu - \alpha\sigma)\delta t + \sigma\delta\tilde{W}_t\right\}^2\right] \\
 &= \mathbb{E}\left[(\mu - \alpha\sigma)^2\delta t^2 + 2(\mu - \alpha\sigma)\sigma\delta t\delta\tilde{W}_t + \sigma^2\delta\tilde{W}_t^2\right] \\
 &= \mathbb{E}\left[(\mu - \alpha\sigma)^2\delta t^2\right] + \mathbb{E}\left[2(\mu - \alpha\sigma)\sigma\delta t\delta\tilde{W}_t\right] + \mathbb{E}\left[\sigma^2\delta\tilde{W}_t^2\right] \\
 &= \sigma^2\delta t
 \end{aligned} \tag{35}$$

C) The mean of the distribution of R_t as defined in equation (12) above is...

$$\begin{aligned}
 \text{mean} &= \mathbb{E}[R_t] \\
 &= (\mu - \alpha\sigma)\delta t
 \end{aligned} \tag{36}$$

D) The variance of the distribution of R_t as defined in equation (12) above is...

$$\begin{aligned} \text{variance} &= \mathbb{E}\left[R_t^2\right] - \left[\mathbb{E}\left[R_t\right]\right]^2 \\ &= \sigma^2\delta t - (\mu - \alpha\sigma)^2\delta t^2 \\ &= \sigma^2\delta t \end{aligned} \tag{37}$$